Volterra Analysis of Spectral Regrowth

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***Abstract*— This letter presents an analysis of the broadening of the bandwidth of narrowband modulated signals, commonly called *spectral regrowth*, caused by weak nonlinearities. The analysis uses Volterra methods and operates entirely in the frequency domain. It is applicable to signals having either discrete or continuous spectra.**

***Index Terms*— Compression, intermodulation distortion, nonlinear distortion, Volterra series.**

I. Introduction

WHEN A modulated signal passes through a weakly nonlinear circuit, its bandwidth is broadened by odd-order nonlinearities. This phenomenon, called *spectral regrowth* or *spectral regeneration*, is caused by the generation of mixing products between the individual frequency components of the spectrum. As such, it is a form of intermodulation distortion and is amenable to analysis by Volterra methods.

Very little previous work has been done on this problem. A recent paper by Sevic and Steer [1] represents the current state of the art, [2] shows an approach using a low-frequency transformation, and [3] presents additional information relevant to a Volterra approach. In contrast, [4] describes a harmonic-balance approach. Of course, considerable work has been done on the more general problem of analyzing intermodulation distortion by Volterra methods [5]–[8].

In the following analysis, we assume that 1) the circuit or system is weakly nonlinear; 2) the percentage bandwidth of the input spectrum is small (on the order of a few percent); and 3) third-order effects dominate. The latter two limitations can be removed, at the expense of increased computational complexity, but the first is a fundamental limitation of Volterra-series analysis.

II. Analysis

Fig. 1 shows the system of interest. It consists of a nonlinear circuit or system having excitation $x(t)$ and response $y(t)$, which in general have frequency spectra. The block is described by its frequency domain, third-order nonlinear transfer function $H_3(\omega_1, \omega_2, \omega_3)$ and, of course, its linear transfer function $H_1(\omega)$.

If $x(t)$ has a discrete spectrum, it can be expressed as a sum of sinusoids

$$x(t) = \frac{1}{2} \sum_{\substack{q=-N \\ q\neq 0}}^{N} X(\omega_q) \exp(j\omega_q t). \tag{1}$$

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Each component of the first-order (linear) output $y_1(t)$ is given by

$$y_1(t) = \frac{1}{2} \sum_{\substack{q=-N \\ q\neq 0}}^{N} H_1(\omega_q) X(\omega_q) \exp(j\omega_q t) \tag{2}$$

and the third-order output $y_3(t)$ is

$$y_3(t) = \frac{1}{8} \sum_{q1=-N}^{N} \sum_{\substack{q2=-N \\ qn\neq 0}}^{N} \sum_{q3=-N}^{N} H_3(\omega_{q1}, \omega_{q2}, \omega_{q3}) \times X(\omega_{q1}) X(\omega_{q2}) X(\omega_{q3}) \exp(\mathrm{j}(\omega_{\mathrm{q1}} + \omega_{\mathrm{q2}} + \omega_{\mathrm{q3}})\mathrm{t}). \tag{3}$$

The third-order component at any frequency $\omega_{q1} + \omega_{q2} + \omega_{q3}$ is found by selecting all the terms in (3) at that frequency. It becomes

$$Y_3(\omega_{q1} + \omega_{q2} + \omega_{q3}) = \frac{t_q}{4} H_3(\omega_{q1}, \omega_{q2}, \omega_{q3}) X(\omega_{q1}) X(\omega_{q2}) X(\omega_{q3}) \tag{4}$$

where t_q is the number of identical terms in the summation (3) at $w_{q1} + w_{q2} + w_{q3}$. The coefficient t_q depends upon both the mixing product of interest and the number of spectral components. It is important to note that many of these components occur at the same frequencies as the components of the input spectrum.

The components that contribute to spectral regrowth are nth order, where n is odd and $(n-1)/2$ mixing-frequency terms are negative. We assume that third-order terms dominate, and one of the mixing frequencies is negative. Thus, we are interested in terms of the form

$$Y_3(\omega_{q1} + \omega_{q2} - \omega_{q3}) = \frac{3t_q}{4} H_3(\omega_{q1}, \omega_{q2}, -\omega_{q3}) X(\omega_{q1}) X(\omega_{q2}) X^*(\omega_{q3}) \tag{5}$$

where the asterisk indicates a complex conjugate. (The coefficient of 3 accounts for the fact that any of the three frequency terms can be negative.) Finally, for narrowband inputs, we can assume that both the linear nonlinear transfer functions are constant, so

$$H_3(\omega_{q1}, \omega_{q2}, -\omega_{q3}) \approx H_3 \tag{6}$$

$$H_1(\omega) \approx H_1 \tag{7}$$

and (5) becomes

$$Y_3(\omega_{q1} + \omega_{q2} - \omega_{q3}) = \frac{3t_q}{4} H_3 X(\omega_{q1}) X(\omega_{q2}) X^*(\omega_{q3}) \tag{8}$$

This simplifying assumption does not limit the accuracy appreciably; it still allows bandwidths of a few percent,

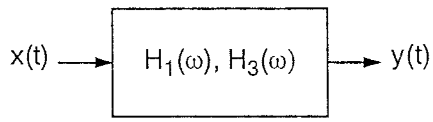

Fig. 1. System of interest. The excitation $x(t)$ and response $y(t)$ can be either voltages or currents. The block can represent either a circuit or a stage in a system.

sometimes more, in practical circuits. For broader-bandwidth waveforms it may be necessary to map the full range of $H_3(\omega_{q1}, \omega_{q2}, -\omega_{q3})$; this is possible, but carries much greater computational cost. If the circuit includes narrowband filtering, the method should be applied to the nonlinear portion alone and the input or output signals filtered appropriately.

III. Practical Implementation

Our in-house Volterra simulator, C/NL2 [9], is used to calculate the third-order nonlinear transfer function of a single-stage field-effect transistor (FET) amplifier circuit. The transfer function is found by exciting the circuit at three frequencies, ω_1, ω_2, and ω_3, and calculating the output voltage at the frequency $\omega_1 + \omega_2 - \omega_3$. It was verified that this transfer function varies in magnitude by only 2.5% and in phase by 1.5 degrees over a 100-MHz bandwidth centered at 10 GHz. The transfer function is

$$H_3 = \frac{V_{o3}}{V_i^3} \tag{9}$$

where V_{o3} is the third-order output voltage and V_i is the voltage of the input tones. The latter are equal in magnitude and have zero phase. Calculating H_3 at 10 frequency points requires only a fraction of 1 s on a 200-MHz P6 computer running Windows NT 4.0.

The evaluation of (8) is performed most easily by general-purpose mathematical-analysis software (we use MathCAD [10]). The input spectrum is read from a file and the third-order products in the vicinity of that spectrum are calculated. These are added to the linear output spectrum. Some of the third-order components are at the same frequency as the linear components; these cause gain compression and AM-to-PM conversion of the linear output. The third-order components that fall outside the linear spectrum, but close to it, represent spectral regrowth. The output spectrum is plotted by MathCAD; this part of the analysis requires approximately 1 s. The resulting linear and distorted spectra are shown in Fig. 2.

IV. Continuous Spectra

Continuous spectra can be handled in much the same way as discrete spectra. For a continuous spectrum, the triple summation in (3) becomes a triple integration. Integrating numerically requires discretizing the spectrum, resulting in an expression very similar to (3).

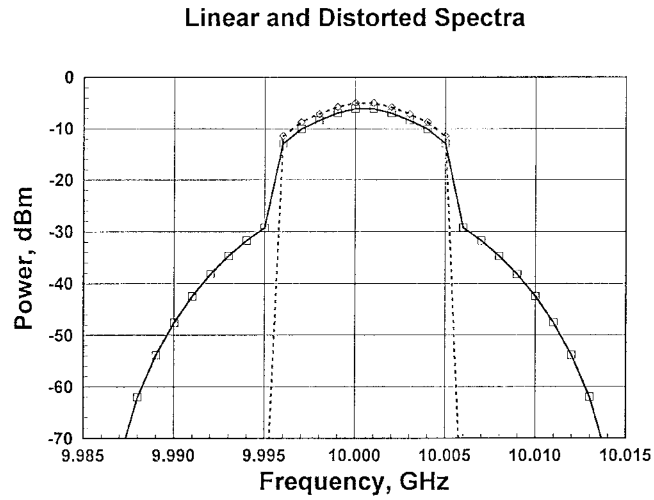

Fig. 2. Output power spectra (dBm) of a FET amplifier operated at the point where individual frequency components are compressed approximately 1 dB. Dashed line: The linear output spectrum, calculated from the linear transfer function alone. Solid line: The output spectrum, compressed and broadened by the circuit's nonlinearities. The markers represent the frequencies of the discrete components.

V. Conclusion

We have shown that spectral regrowth caused by weak nonlinearities can be modeled efficiently by Volterra methods. For narrowband signals it is not necessary to map the full space of the third-order nonlinear transfer function. The method is applicable to either circuits or systems and to signals having either continuous or discrete spectra.

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